

# Violation of an inequality in an experimental test of Leggett's non-local hidden variable theory

## Abstract

I impose previously avoided symmetry conditions in Leggett's non-local hidden variable theory that are required by the conditions in a recent experimental test of this theory [1]. These conditions lead to an inequality for the polarization correlation function that has a maximal violation from the quantum predictions that is 2.5 times larger than reported in this article. Furthermore, when these symmetry conditions are applicable, Leggett's non-local theory also cannot model the observed quantum correlations in previous experiments.

In a recent article entitled *An experimental test of non-local realism* [1], Gröblacher et al. derived an inequality, based on Leggett's non-local hidden variable theory [3], for the mean value  $E$  of the polarization correlation of two photons, and compare it with their data. For a *general* source producing mixtures of two photons, this mean value is determined in Leggett's theory by a distribution function  $F(\mathbf{u}, \mathbf{v})$ , where  $\mathbf{u}$  and  $\mathbf{v}$  are the photon polarization unit vectors on the Poincaré sphere. But according to quantum mechanics, in the reported experiment [1] an entangled two photon state  $\psi_{A,B}$  is created with zero angular momentum and *odd* parity, which therefore has the form (see Appendix) [2]

$$\psi_{A,B} = (1/\sqrt{2})[\psi_A(\mathbf{u})\psi_B(-\mathbf{u}) - \psi_A(-\mathbf{u})\psi_B(\mathbf{u})]. \quad (1)$$

Hence, these two photons always have polarization vectors directed in opposite directions on the Poincarè sphere, i.e.  $\mathbf{v} = -\mathbf{u}$ . Furthermore, this state has the remarkable property that it is *independent* of the orientation of  $\mathbf{u}$  on this sphere (see Appendix). To maintain both of these properties in Leggett's theory requires that the distribution function

$$F(\mathbf{u}, \mathbf{v}) = \frac{1}{4\pi} \delta(\mathbf{u} + \mathbf{v}) \quad (2)$$

where  $\delta$  is the Dirac delta function. Then, the hidden variable polarization correlation function  $E$  depends only on the *relative* orientation of the two polarization analyzers represented by unit vectors  $\mathbf{a}$  and  $\mathbf{b}$  on the Poincarè sphere. Substituting Eq. 2 in Leggett's inequality, equation (4.2) in reference [3] (with an incorrect factor 2 deleted) gives

$$-1 + \frac{1}{4\pi} \int d\mathbf{u} |\mathbf{u} \cdot (\mathbf{a} + \mathbf{b})| \leq \mathbf{P}(\mathbf{a}, \mathbf{b}) \leq 1 - \frac{1}{4\pi} \int d\mathbf{u} |\mathbf{u} \cdot (\mathbf{a} - \mathbf{b})|. \quad (3)$$

The angular integrations on both sides of this inequality can be easily carried out by choosing for the  $z$  axis the directions of the vectors  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$  respectively, and one obtains

$$-1 + |\cos(\phi/2)| \leq -E(\phi) \leq 1 - |\sin(\phi/2)|, \quad (4)$$

where  $\phi$  is the angle between the unit vectors  $\mathbf{a}$  and  $\mathbf{b}$ , and  $E(\phi) = -P((\mathbf{a}, \mathbf{b}))$ . This inequality is more restrictive than the one derived by Gröblacher et al [1].

In quantum mechanics, one finds that  $E$  has the form (see Appendix)

$$E_Q(\phi) = -\mathbf{a} \cdot \mathbf{b} = -\cos(\phi), \quad (5)$$

which is in good agreement with the reported experimental results, but it violates this inequality for  $\phi$  in the ranges  $0^\circ < \phi < 60^\circ$  and  $120^\circ < \phi < 180^\circ$ . The maximal violation occurs at  $\phi = 28.8^\circ$  and  $\phi = 151.2^\circ$ , and its magnitude is 2.5 larger than the value obtained by the inequality of Gröblacher et al. [1]. Our inequality also contradicts their claim that "existing data of all Bell tests cannot be used to test the class of non-local theories consider here".

When the entangled two photon state of zero angular momentum has *even* parity, as is the case, for example, with the photon pairs created in the first experiments testing Bell's inequality [4],[5], I find that (see Appendix)

$$E_Q = \mathbf{a} \cdot \mathbf{b} - 2\mathbf{a}_z \mathbf{b}_z, \quad (6)$$

where the  $z$ -axis in the Poincarè sphere is chosen to represent circular polarization. In the original experiments, the analyzers measured only linear polarization for which  $\mathbf{a}_z = \mathbf{b}_z = \mathbf{0}$ . It would be of interest in this case also to verify experimentally the correlations for elliptical polarization when  $\mathbf{a}_z$  and  $\mathbf{b}_z$  do not vanish.

## Appendix. Two photon polarization correlation function in quantum mechanics

Let  $\psi^+$  represent the state of right handed circular polarization along the  $z$ -axis of the Poincarè sphere, and  $\psi^-$  the corresponding state of left handed circular polarization. Then

$$\psi^\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y) \quad (7)$$

where  $\mathbf{e}_x$  is a unit vector for linear polarization along the  $x$  axis, and  $\mathbf{e}_y$  is the corresponding vector for the  $y$  axis. A general state of elliptic polarization  $\psi(\mathbf{u})$  is given by

$$\psi(\mathbf{u}) = e^{-i\phi/2} \cos(\theta/2) \psi^+ + e^{i\phi/2} \sin(\theta/2) \psi^-, \quad (8)$$

where  $\theta, \phi$  are the polar angles representing the unit vector  $\mathbf{u}$  on the Poincarè sphere. This state is an eigenstate of the Pauli spin matrix  $\sigma \cdot \mathbf{u}$  along the unit vector  $\mathbf{u}$  with eigenvalue 1,

$$\sigma \cdot \mathbf{u} \psi(\mathbf{u}) = \psi(\mathbf{u}), \quad (9)$$

and it has the useful property

$$(\psi(\mathbf{a}) | \sigma \cdot \mathbf{b} | \psi(\mathbf{a})) = \mathbf{a} \cdot \mathbf{b}. \quad (10)$$

The polarization correlation function  $E_{A,B}$  for a general two photon state  $\psi_{A,B}(\mathbf{u}, \mathbf{v})$  is given by

$$E_{A,B} = (\psi_{A,B} | \sigma_A \cdot \mathbf{a} \sigma_B \cdot \mathbf{b} | \psi_{A,B}), \quad (11)$$

where  $\mathbf{a}, \mathbf{b}$  are the elliptical polarization analyzers for photons  $A, B$ . The parity transformation  $P$  is defined by

$$P\psi(\mathbf{u}) = -i\psi(-\mathbf{u}) \quad (12)$$

and  $P^2 = 1$ . Hence, the state  $\psi_{A,B}$  defined by Eq. 1 is a state of odd parity, and substituting the representation Eq. 8 for  $\psi(\mathbf{u})$  into this equation one obtains

$$\psi_{A,B}^{odd} = -\frac{i}{\sqrt{(2)}}(\psi_A^+\psi_B^- - \psi_A^-\psi_B^+) \quad (13)$$

which demonstrates that this state is independent of the direction of the unit vector  $\mathbf{u}$ . Substituting this state in Eq. 11 one obtains

$$E_{A,B}^{odd} = -\mathbf{a} \cdot \mathbf{b}. \quad (14)$$

The corresponding two photon state of even parity is

$$\psi_{A,B}^{even} = -\frac{i}{\sqrt{(2)}}(\psi_A^+\psi_B^- + \psi_A^-\psi_B^+), \quad (15)$$

and substituting in Eq. 11 one obtains

$$E_{A,B}^{even} = \mathbf{a} \cdot \mathbf{b} - 2\mathbf{a}_z \mathbf{b}_z. \quad (16)$$

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## References

- [1] S. Gröblacher, T. Paterek, R. Kaltenbaek, C. Brukner, M. Zukowski, M. Aspelmeyer & A. Zeilinger,  
*An experimental test of non-local realism.*  
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- [2] In reference [1], under the caption of Figure 2, the authors report that in their experiment “ the crystal is aligned to produce the polarization entangled singlet state”. This state is written in a basis for circularly polarized states aligned along the z-axis of their Poincarè sphere, which corresponds to Eq. 13 in the Appendix of this paper . This particular representation, however, does not reveal the symmetry property of this state which is shown in this Appendix to be independent of the choice of polarization of the basis state.

- [3] A.J. Leggett, *Nonlocal Hidden-Variable Theories and Quantum Mechanics: An Incompatibility Theorem*.  
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 My inequality Eq. 3 is obtained from Eq. 4.2 in this paper, but with the erroneous factor 2 which appears on both sides of this equation omitted.  
 See also A.J. Leggett, Erratum, *Foundations of Physics*, to be published.
- [4] S.J. Freedman and J.F. Clauser, *Experimental Test of Local Hidden Variable Theories*.  
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- [5] A. Aspect, P. Grangier, and G. Roger, *Experimental Tests of Realistic Local Theories via Bell's Theorem*.  
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